

Morrey-Campanato estimates for the moments of stochastic integral operators and its application to SPDEs

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Abstract

In this paper, we are concerned with the estimates for the moments of stochastic convolution integrals. We first deal with the stochastic singular integral operators and we aim to derive the Morrey-Campanato estimates for the p -moments (for $p \geq 1$). Then, by utilising the embedding theory between the Campanato space and Hölder space, we establish the norm of $C^{\theta, \theta/2}(\bar{D})$, where $\theta \geq 0$, $\bar{D} = \bar{G} \times [0, T]$ for arbitrarily fixed $T \in (0, \infty)$ and $G \subset \mathbb{R}^d$. As an application, we consider the following stochastic (fractional) heat equations with additive noises

$$du_t(x) = \Delta^\alpha u_t(x)dt + g(t, x)d\eta_t, \quad u_0 = 0, \quad 0 \leq t \leq T, x \in G,$$

where $\Delta^\alpha = -(-\Delta)^\alpha$ with $0 < \alpha \leq 1$ (the fractional Laplacian), $g : [0, T] \times G \times \Omega \rightarrow \mathbb{R}$ is a joint measurable coefficient, and $\eta_t, t \in [0, T]$, is either the Brownian motion or a Lévy process on a given filtered probability space $(\Omega, \mathcal{F}, P; \{\mathcal{F}_t\}_{t \in [0, T]})$. The Schauder estimate for the p -moments of the solution of the above equation is obtained. The novelty of the present paper is that we obtain the Schauder estimate for parabolic stochastic partial differential equations with Lévy noise.

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1 Introduction

For a stochastic process $\{X_t, t \in [0, T]\}$, there are two important aspects worth investigating. One is the associated probability density functions (PDFs) or its probability laws, and the other is the moments estimate. But for a stochastic process depending on spatial variable (to be more precise, a random field), that is, $X_t = X(t, x, \omega)$ with x standing for a spatial variable, it would be hard to consider its PDFs or probability laws. Fortunately, we could get some moments estimate. In this

paper, we focus on the estimates for solutions of (parabolic) stochastic partial differential equations (SPDEs), in particular, on the Schauder estimate for the SPDEs.

For (parabolic) SPDEs, certain kinds of estimates for the solutions have been well studied. By using parabolic Littlewood-Paley inequality, Krylov [23] proved that for SPDEs of the following type

$$du = \Delta u dt + g dw_t, \quad (1.1)$$

it holds that

$$\mathbb{E} \|\nabla u\|_{L^p((0,T) \times \mathbb{R}^d)}^p \leq C(d, p) \mathbb{E} \|g\|_{L^p((0,T) \times \mathbb{R}^d)}^p, \quad (1.2)$$

where w_t is a Wiener process and $p \in [2, \infty)$. Moreover, van Neerven et al. [29] made a significant extension of (1.2) to a class of operators A which admit a bounded H^∞ -calculus of angle less than $\pi/2$. Kim [16] established a BMO estimate for stochastic singular integral operators. And as an application, he considered (1.1) and interestingly he obtained the q -th order BMO quasi-norm of the derivative of u is controlled by $\|g\|_{L^\infty}$. More recently, Kim et al. [18] studied the parabolic Littlewood-Paley inequality for a class of time-dependent pseudo-differential operators of arbitrary order, and applied their result to a high-order stochastic PDE. We refer the interested readers to [4, 12] for a comprehensive account on the BMO estimates.

Recently, Yang [31] considered the following SPDEs

$$du = \Delta^{\frac{\alpha}{2}} u dt + f dX_t, \quad u_0 = 0, \quad 0 < t < T,$$

where $\Delta^{\frac{\alpha}{2}} = -(-\Delta)^{\frac{\alpha}{2}}$, for $0 < \alpha < 2$, are nothing but the fractional Laplace operators, and X_t is a Lévy process. The author obtained a parabolic Triebel-Lizorkin space estimate for the convolution operator.

In our paper [26], we consider the stochastic singular operator

$$\begin{aligned} \mathcal{G}g(t, x) &= \int_0^t \int_Z K(t, s, \cdot) * g(s, \cdot, z)(x) \tilde{N}(dz, ds) \\ &= \int_0^t \int_Z \int_{\mathbb{R}^d} K(t - s, x - y) g(s, y, z) dy \tilde{N}(dz, ds), \end{aligned} \quad (1.3)$$

for $g : [0, T] \times \mathbb{R}^d \times Z \times \Omega \rightarrow \mathbb{R}$ being a predictable process, where \tilde{N} is a compensated Poisson measure. Under appropriate conditions on the kernel K , we obtained the q -th order BMO estimate. As an application, we obtained the q -th order BMO estimate for the solution of the stochastic nonlocal heat equation.

For the regularity of SPDEs, several important works have been established, see [20, 21, 24, 29, 32]. Similar to the regularity of PDE, the regularity of SPDEs can be divided into two parts. One is the L^p -theory. Krylov [24] obtained the L^p -theory of SPDEs on the whole space. Later, Kim [20, 21] established the L^p -theory of SPDEs on the bounded domain. Using the Moser's iteration scheme, Denis et al. [10] also obtained the L^p -theory of SPDEs on the bounded domain. The other part is the Schauder estimates. Debussche et al. [8] proved that the solution of SPDEs is Hölder continuous in both time and space variables. Du-Liu [11] established the $C^{2+\alpha}$ -theory for SPDEs on the whole space. Hsu-Wang [13] used stochastic De Giorgi iteration technique to prove that the solutions of SPDEs are almost surely Hölder continuous in both space and time variables.

The above mentioned results about the regularity of the solutions of SPDEs belongs to the space $L^p(\Omega; C^{\alpha, \beta}([0, T] \times G))$, where G is a bounded domain in \mathbb{R}^d . Now, there is an natural question, that is, can one get the Hölder estimate for the p -moment? In other words, can we derive

the estimate in $C^{\alpha,\beta}([0, T] \times G; L^p(\Omega))$? We note that Du-Liu [11] obtained the $C^{2+\alpha}$ -theory for SPDEs in $C^{\alpha,\beta}([0, T] \times G; L^p(\Omega))$, where the Dini continuous is needed for the stochastic term. The method used in [11] is the Sobolev embedding theorem and the iteration technique under the condition that the noise term satisfies Dini continuity. In this paper, we would like to consider a simple case, that is, the equation with additive noise. We first derive the Morrey-Campanato estimates for the stochastic convolution operators and then, by utilising the embedding theorem between Campanato space and Hölder space, we establish the norm of $C^{\theta,\theta/2}$. As an application, we show that the solutions of partial differential equations driven by Brownian motion or by Lévy noise are Hölder continuous in the both time and space variables on the whole space. The novelty in our present paper is the approach we used is different from those in [10, 11, 13]. We would like to point out that by using the Morrey-Campanato estimates and the embedding theorem, the Hölder estimates can be easily derived, and on the other hand our Morrey-Campanato estimates can be obtained by directly calculation, thus our method is indeed simpler than others, see [25, Lemma 4.3] for the deterministic parabolic equations. Besides, we establish the Schauder estimates for the solutions of partial differential equations driven by Lévy noise.

The paper is organized as follows. In Section 2, the next section, we set up our main results and present corresponding proofs. Section 3, the final section, gives an application of our results.

Before ending up this section, let us introduce some notations. As usual, \mathbb{R}^d stands for the d -dimensional Euclidean space of points $x = (x_1, \dots, x_d)$ with $|\cdot|$ being its usual Euclidean norm, and $B_r(x) := \{y \in \mathbb{R}^d : |x - y| < r\}$ as well as $B_r := B_r(0)$. We use \mathbb{R}_+ to denote the set $\{x \in \mathbb{R} : x > 0\}$, $a \wedge b := \min\{a, b\}$, $a \vee b := \max\{a, b\}$ and $L^p := L^p(\mathbb{R}^d)$. Finally, we write $N = N(a, b, \dots)$ for a constant N which depends on a, b, \dots .

2 Main Results

We first recall some definitions and known results. Set, for $X = (t, x) \in \mathbb{R} \times \mathbb{R}^d$ and $Y = (s, y) \in \mathbb{R} \times \mathbb{R}^d$, the following

$$\delta(X, Y) := \max \left\{ |x - y|, |t - s|^{\frac{1}{2}} \right\}.$$

Let $Q_c(X)$ be the ball centered in $X = (t, x)$ and of radius c , i.e.,

$$\begin{aligned} Q_c(X) &:= \{Y = (s, y) \in \mathbb{R} \times \mathbb{R}^d : \delta(X, Y) < c\} \\ &= (t - c^2, t + c^2) \times B_c(x). \end{aligned}$$

Fix $T \in (0, \infty]$ arbitrarily. Denote

$$\mathcal{O}_T = (0, T) \times \mathbb{R}^d.$$

Let D be a bounded domain in \mathbb{R}^{d+1} and for $X \in D$, $D(X, r) := D \cap Q_r(X)$ and $d(D) := \text{diam} D$. We first introduce the definition of Campanato space.

Definition 2.1 (*Campanato Space*) Let $p \geq 1$ and $\theta \geq 0$. u belongs to Campanato space $\mathcal{L}^{p,\theta}(D; \delta)$ if $u \in L^p(D)$ and

$$[u]_{\mathcal{L}^{p,\theta}(D; \delta)} := \left(\sup_{X \in D, d(D) \geq \rho > 0} \frac{1}{|D(X, \rho)|^\theta} \int_{D(X, \rho)} |u(Y) - u_{X, \rho}|^p dY \right)^{1/p} < \infty,$$

and

$$\|u\|_{\mathcal{L}^{p,\theta}(D; \delta)} := \left(\|u\|_{L^p(D)}^p + [u]_{\mathcal{L}^{p,\theta}(D; \delta)}^p \right)^{1/p},$$

where $|D(X, \rho)|$ stands for the Lebesgue measure of $D(X, \rho)$ and

$$u_{X, \rho} = \frac{1}{|D(X, \rho)|} \int_{D(X, \rho)} u(Y) dY.$$

It is easy to verify that Campanato space is a Banach space and has the following property (see Appendix): if $1 \leq p \leq q < \infty$, $(\theta - p)/p \leq (\sigma - p)/q$, it holds that

$$\mathcal{L}^{q, \sigma}(D; \delta) \subset \mathcal{L}^{p, \theta}(D; \delta).$$

Next we recall the definition of Hölder space.

Definition 2.2 (*Hölder Space*) Let $0 < \alpha \leq 1$. u belongs to Hölder space $C^\alpha(\bar{D}; \delta)$ if u satisfies

$$[u]_{C^\alpha(\bar{D}; \delta)} := \sup_{X \in D, d(D) \geq \rho > 0} \frac{|u(X) - u(Y)|}{\delta(X, Y)^\alpha} < \infty,$$

and

$$\|u\|_{C^\alpha(\bar{D}; \delta)} := \sup_D |u| + [u]_{C^\alpha(\bar{D}; \delta)}.$$

Definition 2.3 Let $D \subset \mathbb{R}^{d+1}$. Domain D is called A -type if there exists a constant $A > 0$ such that $\forall X \in D$, $0 < \rho \leq \text{diam} D$, it holds that

$$|D(X, \rho)| \geq A |Q_\rho(X)|.$$

Comparing with the two space, we have the following relations.

Proposition 2.1 Assume that D is an A -type bounded domain. Then we have the following relation: when $1 < \theta \leq 1 + \frac{p}{d+2}$ and $p \geq 1$,

$$\mathcal{L}^{p, \theta}(D; \delta) \cong C^\alpha(\bar{D}; \delta),$$

where

$$\alpha = \frac{(d+2)(\theta-1)}{p},$$

where d is the dimension of the space.

Here $A \cong B$ means that both $A \subseteq B$ and $B \subseteq A$ hold. We will obtain the Campanato estimates under some assumptions on the kernel K . Noting that

$$\begin{aligned} & \left(\sup_{X \in D, d(D) \geq \rho > 0} \frac{1}{|D(X, \rho)|^\theta} \int_{D(X, \rho)} |u(Y) - u_{X, \rho}|^p dY \right)^{1/p} \\ &= \left(\sup_{X \in D, d(D) \geq \rho > 0} \frac{1}{|D(X, \rho)|^\theta} \int_{D(X, \rho)} \left| u(Y) - \frac{1}{|D(X, \rho)|} \int_{D(X, \rho)} u(Z) dZ \right|^p dY \right)^{1/p} \\ &\leq \left(\sup_{X \in D, d(D) \geq \rho > 0} \frac{1}{|D(X, \rho)|^{1+\theta}} \int_{D(X, \rho)} \int_{D(X, \rho)} |u(Y) - u(Z)|^p dZ dY \right)^{1/p}, \end{aligned}$$

so the definition of semi-norm of the Campanato space can be replaced by the above inequality. We also remark that in order to get the Hölder estimate, the range of θ must be larger than 1.

Now, we talk about two spaces $L^p(\Omega; \mathcal{L}^{p,\theta}(D; \delta))$ and $\mathcal{L}^{p,\theta}((D; \delta); L^p(\Omega))$. If we want to prove $u \in L^p(\Omega; \mathcal{L}^{p,\theta}(D; \delta))$, that is,

$$\mathbb{E}[u]_{\mathcal{L}^{p,\theta}(D; \delta)}^p := \mathbb{E} \sup_D \frac{1}{|D|^{1+\theta}} \int_D \int_D |h(t, x) - h(s, y)|^p dt dx ds dy < \infty,$$

our first idea is to prove the two maps \mathbb{E} and $\sup_{t,x}$ can be interchanged. Unfortunately, it is hard to give a sufficient condition to assure the above idea holds. The second idea is to prove the norm of u in $\mathcal{L}^{p,\theta}(D; \delta)$ is bounded almost surely. The two ideas is hard to come true. And thus we must adjust our idea. We also remark that the mean of the space $\mathcal{L}^{p,\theta}((D; \delta); L^p(\Omega))$ is that we call

$$u \in \mathcal{L}^{p,\theta}((D; \delta); L^p(\Omega)), \quad \text{if} \quad \|u\|_{L^p(\Omega)} \in \mathcal{L}^{p,\theta}(D; \delta).$$

In other words, the following norm is finite

$$\begin{aligned} & [\|u\|_{L^p(\Omega)}]_{\mathcal{L}^{p,\theta}(D; \delta)}^p \\ &:= \sup_D \frac{1}{|D|^{1+\theta}} \int_D \int_D \left| \|u\|_{L^p(\Omega)}(t, x) - \|u\|_{L^p(\Omega)}(s, y) \right|^p dt dx ds dy \\ &< \infty. \end{aligned}$$

Using triangular inequality, we have

$$\begin{aligned} & [\|u\|_{L^p(\Omega)}]_{\mathcal{L}^{p,\theta}(D; \delta)}^p \\ &\leq \sup_D \frac{1}{|D|^{1+\theta}} \int_D \int_D \|u(t, x) - u(s, y)\|_{L^p(\Omega)}^p dt dx ds dy \\ &= \sup_D \frac{1}{|D|^{1+\theta}} \mathbb{E} \int_D \int_D |u(t, x) - u(s, y)|^p dt dx ds dy. \end{aligned}$$

Thus we only need to prove that

$$\sup_D \frac{1}{|D|^{1+\theta}} \mathbb{E} \int_D \int_D |u(t, x) - u(s, y)|^p dt dx ds dy < \infty.$$

2.1 Brownian Motion Case

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space endowed with $\{\mathcal{F}_t\}_{t \in [0, T]}$, a filtration on Ω containing all P -null subsets of Ω . Let W_t be a one-dimensional $\{\mathcal{F}_t\}_{t \in [0, T]}$ -adapted Wiener processes defined on $(\Omega, \mathcal{F}, \mathbb{P})$.

For a measurable function h on $\Omega \times \mathcal{O}_T$, we define the stochastic Campanato quasi-norm of h on $\Omega \times \mathcal{O}_T$ as follows:

$$[h]_{\mathcal{L}^{p,\theta}((Q; \delta); L^p(\Omega))}^p := \sup_Q \frac{1}{|Q|^{1+\theta}} \mathbb{E} \int_Q \int_Q |h(t, x) - h(s, y)|^p dt dx ds dy,$$

where the sup is taken over all $Q = D \cap Q_c$ of the type

$$Q_c(t_0, x_0) := (t_0 - c^2, t_0 + c^2) \times B_c(x_0) \subset \mathcal{O}_T, \quad c > 0, t_0 > 0.$$

It is remarked that when $\theta = 1$, this is equivalent to the classical BMO semi-norm which is introduced in John-Nirenberg [15]. If the stochastic Campanato quasi-norm of h is finite, we then say that h belongs to the space $\mathcal{L}^{p,\theta}((Q; \delta); L^p(\Omega))$.

We first consider the Brownian motion case. Given a deterministic kernel $K : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$, we denote for any no-random (i.e., not randomly dependent) $g : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ the following stochastic convolution

$$\mathcal{K}g(t, x) := \int_0^t \int_{\mathbb{R}^d} K(t-r, y) g(r, x-y) dy dW(r). \quad (2.1)$$

Then we have the following result.

Theorem 2.1 *Let D be an A -type bounded domain in \mathbb{R}^{d+1} such that $\bar{D} \subset \mathcal{O}_T$. Suppose that $g \in C^\beta(\mathbb{R}_+ \times \mathbb{R}^d)$, $0 < \beta < 1$, is a non-random function and $g(0, 0) = 0$. Assume that there exists positive constants γ_i ($i = 1, 2$) such that the non-random kernel function satisfies that for any $t \in (0, T]$*

$$\int_0^s \left(\int_{\mathbb{R}^d} |K(t-r, z) - K(s-r, z)|(1 + |z|^\beta) dz \right)^2 dr \leq N(T, \beta)(t-s)^{\gamma_1}, \quad (2.2)$$

$$\int_0^s \left(\int_{\mathbb{R}^d} |K(s-r, z)| dz \right)^2 dr \leq N_0, \quad (2.3)$$

$$\int_s^t \left(\int_{\mathbb{R}^d} |K(t-r, z)|(1 + |z|^\beta) dz \right)^2 dr \leq N(T, \beta)(t-s)^{\gamma_2}, \quad (2.4)$$

where N_0 is a positive constant. Then we have, for $p \geq 1$ and $\beta < \gamma$,

$$[\mathcal{K}g]_{\mathcal{L}^{p, \theta}((D; \delta); L^p(\Omega))} \leq N(N_0, \beta, T, d, p),$$

where $\theta = 1 + \frac{\gamma p}{d+2}$ and $\gamma = \min\{\gamma_1, \gamma_2, \beta\}$.

Proof. Let $(t_0, x_0) \in D \subset \mathcal{O}_T$ and

$$Q_c(t_0, x_0) = (t_0 - c^2, t_0 + c^2) \times B_c(x_0).$$

Then set $C_1 := \text{diam} D$, we have $\bar{D} \subset Q_{C_1}(t_0, x_0)$. Denote $Q := D \cap Q_c(t_0, x_0)$.

Set $t > s$. By the BDG inequality, we have

$$\begin{aligned} & \mathbb{E} \int_Q \int_Q |\mathcal{K}g(t, x) - \mathcal{K}g(s, y)|^p dt dx ds dy \\ = & \mathbb{E} \int_Q \int_Q \left| \int_0^t \int_{\mathbb{R}^d} K(t-r, z) g(r, x-z) dz dW(r) \right. \\ & \left. - \int_0^s \int_{\mathbb{R}^d} K(s-r, z) g(r, y-z) dz dW(r) \right|^p dt dx ds dy \\ \leq & 2^{p-1} \mathbb{E} \int_Q \int_Q \left| \int_0^s \int_{\mathbb{R}^d} (K(t-r, z) - K(s-r, z)) g(r, x-z) dz dW(r) \right|^p \\ & + 2^{p-1} \mathbb{E} \int_Q \int_Q \left| \int_0^s \int_{\mathbb{R}^d} K(s-r, z) (g(r, x-z) - g(r, y-z)) dz dW(r) \right|^p \\ & + 2^{p-1} \mathbb{E} \int_Q \int_Q \left| \int_s^t \int_{\mathbb{R}^d} K(t-r, z) g(r, x-z) dz dW(r) \right|^p dt dx ds dy \\ \leq & N(p) \int_Q \int_Q \left(\int_0^s \int_{\mathbb{R}^d} |K(t-r, z) - K(s-r, z)| |g(r, x-z)|^2 dz dr \right)^{\frac{p}{2}} \\ & + N(p) \int_Q \int_Q \left(\int_0^s \int_{\mathbb{R}^d} |K(s-r, z)| |g(r, x-z) - g(r, y-z)|^2 dz dr \right)^{\frac{p}{2}} \\ & + N(p) \int_Q \int_Q \left(\int_s^t \int_{\mathbb{R}^d} |K(t-r, z) g(r, x-z)|^2 dz dr \right)^{\frac{p}{2}} \\ =: & I_1 + I_2 + I_3. \end{aligned}$$

Estimate of I_1 . By using the Hölder continuous of g , i.e.,

$$\begin{aligned} |g(r, x - z) - g(0, 0)| &\leq C_g \max \left\{ r^{\frac{1}{2}}, |x - z| \right\}^{\beta} \\ &\leq N(g, \beta) (T^{\frac{\beta}{2}} + |x - x_0|^{\beta} + |x_0|^{\beta} + |z|^{\beta}) \\ &\leq N(g, \beta) (T^{\frac{\beta}{2}} + c^{\beta} + |x_0|^{\beta} + |z|^{\beta}), \end{aligned}$$

and (2.2), we have

$$\begin{aligned} I_1 &= N(p) \int_Q \int_Q \left(\int_0^s \left| \int_{\mathbb{R}^d} |K(t - r, z) - K(s - r, z)| |g(r, x - z)| dz \right|^2 dr \right)^{\frac{p}{2}} \\ &\leq N(p, \beta) \int_Q \int_Q \left(\int_0^s \left| \int_{\mathbb{R}^d} |K(t - r, z) - K(s - r, z)| (T^{\frac{\beta}{2}} + c^{\beta} + |x_0|^{\beta} + |z|^{\beta}) dz \right|^2 dr \right)^{\frac{p}{2}} \\ &\leq N(p, \beta, T, x_0) \int_Q \int_Q \left(\int_0^s \left| \int_{\mathbb{R}^d} |K(t - r, z) - K(s - r, z)| (1 + |z|^{\beta}) dz \right|^2 dr \right)^{\frac{p}{2}} \\ &\quad + c^{\beta} N(p, \beta) \int_Q \int_Q \left(\int_0^s \left| \int_{\mathbb{R}^d} |K(t - r, z) - K(s - r, z)| dz \right|^2 dr \right)^{\frac{p}{2}} \\ &\leq N(p, \beta, T, x_0) (1 + |x - y|^{\beta p}) (t - s)^{\frac{\gamma_1 p}{2}} |Q|^2. \end{aligned}$$

The condition (2.3) and

$$|g(r, x - z) - g(r, y - z)| \leq C_g |x - y|^{\beta}$$

imply the following derivation

$$\begin{aligned} I_2 &= N(p) \int_Q \int_Q \left(\int_0^s \left| \int_{\mathbb{R}^d} |K(s - r, z)| |g(r, x - z) - g(r, y - z)| dz \right|^2 dr \right)^{\frac{p}{2}} \\ &\leq N(p, g) \int_Q \int_Q \left(\int_0^s \left| \int_{\mathbb{R}^d} |K(r, z)| |x - y|^{\beta} dz \right|^2 dr \right)^{\frac{p}{2}} \\ &\leq N(N_0, p, g, \beta) |x - y|^{\beta p} |Q|^2. \end{aligned}$$

Estimate of I_3 . By using the property $g(0, 0) = 0$ and (2.4), we get

$$\begin{aligned} I_3 &= N(p) \int_Q \int_Q \left(\int_s^t \left| \int_{\mathbb{R}^d} K(t - r, z) g(r, x - z) dz \right|^2 dr \right)^{\frac{p}{2}} \\ &\leq \int_Q \int_Q \left(\int_s^t \left| \int_{\mathbb{R}^d} |K(r, z)| (T + |x - x_0|^{\beta} + |x_0|^{\beta} + |z|^{\beta}) dz \right|^2 dr \right)^{\frac{p}{2}} \\ &\leq N(p, T, x_0, \beta) \int_Q \int_Q \left(\int_s^t \left| \int_{\mathbb{R}^d} |K(t - r, z)| (1 + |z|^{\beta}) dz \right|^2 dr \right)^{\frac{p}{2}} \\ &\quad + N(p, T, \beta) |x - y|^{\beta p} \int_Q \int_Q \left(\int_s^t \left| \int_{\mathbb{R}^d} |K(t - r, z)| dz \right|^2 dr \right)^{\frac{p}{2}} \\ &\leq N(p, T, x_0, \beta) |Q|^2 (t - s)^{\frac{\gamma_2 p}{2}} (1 + |x - y|^{\beta p}). \end{aligned}$$

Noting that $(t, x) \in Q_c$ and $(s, y) \in Q_c$, we have

$$0 \leq t - s \leq 2c^2 \quad \text{and} \quad |x - y| \leq |x - x_0| + |y - x_0| \leq 2c.$$

Using the above inequality and the properties of A -type domain, we deduce

$$\begin{aligned} I_1 &\leq N(p, T, \beta, x_0)(1 + c^{\beta p})c^{\gamma_1 p}|Q|^2; \\ I_2 &\leq N(N_0, p, g, \beta)c^{\beta p}|Q|^2; \\ I_3 &\leq N(p, T, x_0, \beta)|Q|^2c^{\gamma_2 p}(1 + c^{\beta p}). \end{aligned}$$

Combining the estimates of I_1, I_2 and I_3 , we get

$$\begin{aligned} &\mathbb{E} \int_Q \int_Q |u(t, x) - u(s, y)|^p dt dx ds dy \\ &\leq N(\beta, N_0, T, p)|Q|^2(c^{\beta p} + 1)(c^{\beta p} + c^{\gamma_1 p} + c^{\gamma_2 p}). \end{aligned}$$

Since D is a A -type bounded domain, we have $c \leq \text{diam} D$ and

$$A|Q_c(t_0, x_0)| \leq |Q| \leq |Q_c(t_0, x_0)|.$$

We remark that $|Q_c(t_0, x_0)| = Nc^{d+2}$ and $0 < \beta \leq 1$, where N is a positive constant which does not depend on c . Noting that $Q \subset Q_{C_1}$, we have

$$\begin{aligned} &\mathbb{E} \int_Q \int_Q |u(t, x) - u(s, y)|^p dt dx ds dy \\ &\leq N(\beta, N_0, C_1, d, T)|Q|^{2+\frac{\gamma p}{d+2}}, \end{aligned}$$

where $\gamma = \min\{\gamma_1, \gamma_2, \beta\}$, which yield that

$$\begin{aligned} &[\mathcal{K}g]_{\mathcal{L}^{p, \theta}(D; \delta)} \\ &= \sup_Q \frac{1}{|Q|^{1+\theta}} \mathbb{E} \int_Q \int_Q |\mathcal{K}g(t, x) - \mathcal{K}g(s, y)|^p dt dx ds dy \\ &\leq N(\beta, N_0, T, d, p), \end{aligned}$$

where $\theta = 1 + \frac{\gamma p}{d+2}$. The proof of Theorem 2.1 is complete. \square

Theorem 2.1 shows that $\mathcal{K}g(t, x) \in \mathcal{L}^{p, \theta}((Q; \delta); L^p(\Omega))$. That is, $\|\mathcal{K}g\|_{L^p(\Omega)} \in \mathcal{L}^{p, \theta}(Q; \delta)$. Applying the result of Proposition 2.1, we have the following result.

Corollary 2.1 *Assume all the assumptions in Theorem 2.1 hold, then*

$$\mathcal{K}g(t, x) \in C^\gamma((\bar{D}; \delta); L^p(\Omega)).$$

Remark 2.1 1. It follows from Theorem 2.1 and Corollary 2.1 that $\mathcal{K}g(t, x) \in C^\gamma((\bar{D}; \delta); L^p(\Omega))$ and $\gamma = \min\{\gamma_1, \gamma_2, \beta\}$ if $g \in C^\beta(\mathbb{R}_+ \times \mathbb{R}^d)$ and $g(0, 0) = 0$. For special kernel, we can let $\gamma = \beta$, see Theorem 3.1. That is to say, the regularity of $\mathcal{K}g(t, x)$ depends heavily on the noise term g .

2. It is easy to prove that if $g \in C^{k+\beta, \beta/2}(\mathbb{R}_+ \times \mathbb{R}^d)$ and $\nabla^k g(0, 0) = 0$, then $\mathcal{K}g(t, x) \in C^{k+\gamma, \gamma/2}(\bar{D}; \delta)$ under the assumptions of Theorem 2.1. Here $g \in C^{k+\beta, \beta/2}(\mathbb{R}_+ \times \mathbb{R}^d)$ denotes that the k -order of g w.r.t spatial variable belongs to C^β , and that g w.r.t time variable belongs to $C^{\beta/2}$.

3. The regularity w.r.t time variable can not be improved because of the fact that the regularity of Brownian motion w.r.t time variable is $C^{\frac{1}{2}-}$.

4. If the kernel function K is random, the similar result also holds. The constant N in Theorem 2.1 depending on the choice of x_0 can be removed provided that

$$\mathbb{E} \left[\|g\|_{L^\infty(\mathcal{O}_T)}^{p_0} \right] < \infty,$$

where $p_0 \geq 1$ and $1 \leq p \leq p_0$.

5. The method used in Theorem 2.1 is similar to that in [25] for the interior Schauder estimate, see [25, Lemma 4.3].

In Theorem 2.1, the noise term g depends on the times and spatial variables. A natural question is: if g does not depend on the time t , the result of Theorem 2.1 will also hold? Next, we answer this question.

Theorem 2.2 *Suppose that $g \in C^\beta(\mathbb{R}^{d+1})$, $0 < \beta < 1$ and $g(0) = 0$. Assume further that (2.2)-(2.4) hold. Let D be a A -type bounded domain in \mathbb{R}^{d+1} such that $\bar{D} \subset \mathcal{O}_T$. Then we have, for $p \geq 1$,*

$$[\mathcal{K}g]_{\mathcal{L}^{p,\theta}(D;\delta)} \leq N(N_0, \beta, T, d, p),$$

where $\theta = 1 + \frac{\gamma p}{d+2}$ and $\gamma = \min\{\gamma_1, \gamma_2, \beta\}$.

Proof. The definition of Q is the same as in the proof of Theorem 2.1. Fix $t > s$. The BDG inequality implies that

$$\begin{aligned} & \mathbb{E} \int_Q \int_Q |\mathcal{K}g(t, x) - \mathcal{K}g(s, y)|^p dt dx dy \\ = & \mathbb{E} \int_Q \int_Q \left| \int_0^t \int_{\mathbb{R}^d} K(t-r, z) g(x-z) dz dW(r) \right. \\ & \left. - \int_0^s \int_{\mathbb{R}^d} K(s-r, z) g(y-z) dz dW(r) \right|^p dt dx dy \\ \leq & N(p) \mathbb{E} \int_Q \int_Q \left| \int_0^s \int_{\mathbb{R}^d} (K(t-r, z) - K(s-r, z)) g(y-z) dz dW(r) \right|^p \\ & + N(p) \mathbb{E} \int_Q \int_Q \left| \int_0^s \int_{\mathbb{R}^d} K(t-r, z) (g(x-z) - g(y-z)) dz dW(r) \right|^p \\ & + N(p) \mathbb{E} \int_Q \int_Q \left| \int_s^t \int_{\mathbb{R}^d} K(t-r, z) g(x-z) dz dW(r) \right|^p dt dx dy \\ \leq & N(p) \mathbb{E} \int_Q \int_Q \left(\int_0^s \left| \int_{\mathbb{R}^d} (K(t-r, z) - K(s-r, z)) g(y-z) dz \right|^2 dr \right)^{\frac{p}{2}} \\ & + N(p) \mathbb{E} \int_Q \int_Q \left(\int_0^s \left| \int_{\mathbb{R}^d} K(t-r, z) (g(x-z) - g(y-z)) dz \right|^2 dr \right)^{\frac{p}{2}} \\ & + N(p) \mathbb{E} \int_Q \int_Q \left(\int_s^t \left| \int_{\mathbb{R}^d} K(t-r, z) g(x-z) dz \right|^2 dr \right)^{\frac{p}{2}} \\ =: & I_1 + I_2 + I_3. \end{aligned}$$

Noting again that $(t, x) \in Q_c$ and $(s, y) \in Q_c$, we have

$$0 \leq t - s \leq 2c^2 \quad \text{and} \quad |x - y| \leq |x - x_0| + |y - x_0| \leq 2c.$$

The Hölder continuous of g and (2.2)-(2.4) give that

$$I_1 + I_2 + I_3 \leq N(N_0, p, \beta, T, d) |Q|^2 (c^{\beta p} + c^{\gamma_1 p} + c^{\gamma_2 p}),$$

which implies the desired result. The proof is complete. \square

Remark 2.2 *By using Proposition 2.1, one can get $\mathcal{K}g(t, x) \in C^\gamma((\bar{D}; \delta); L^p(\Omega))$. In particular, taking $g = \text{constant}$, we have the regularity of time variable is $C^{\frac{1}{2}-}$ and the regularity of spatial variable is C^∞ .*

2.2 Lévy Noise Case

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete probability space such that $\{\mathcal{F}_t\}_{t \in [0, T]}$ is a filtration on Ω containing all P -null subsets of Ω and \mathbb{F} be the predictable σ -algebra associated with the filtration $\{\mathcal{F}_t\}_{t \in [0, T]}$. We are given a σ -finite measure space (Z, \mathcal{Z}, ν) and a Poisson random measure μ on $[0, T] \times Z$, defined on the stochastic basis. The compensator of μ is $\text{Leb} \otimes \nu$, and the compensated martingale measure $\tilde{N} := \mu - \text{Leb} \otimes \nu$.

In this subsection, we consider the stochastic singular integral operator

$$\begin{aligned} \mathcal{G}g(t, x) &= \int_0^t \int_Z K(t, s, \cdot) * g(s, \cdot, z)(x) \tilde{N}(dz, ds) \\ &= \int_0^t \int_Z \int_{\mathbb{R}^d} K(t-s, x-y) g(s, y, z) dy \tilde{N}(dz, ds) \end{aligned} \quad (2.5)$$

for \mathbb{F} -predictable processes $g : [0, T] \times \mathbb{R}^d \times Z \times \Omega \rightarrow \mathbb{R}$. For simplicity, we assume that the kernel function is deterministic. We first recall the Kunita's first inequality.

Definition 2.4 (*Kunita's first inequality [1, Theorem 4.4.23]*) *For any $p \geq 2$, there exists $N(p) > 0$ such that*

$$\begin{aligned} \mathbb{E} \left(\sup_{0 \leq t \leq T} |I(t)|^p \right) &\leq N(p) \left\{ \mathbb{E} \left[\left(\int_0^T \int_Z |H(t, z)|^2 \nu(dz) dt \right)^{p/2} \right] \right. \\ &\quad \left. + \mathbb{E} \left[\int_0^T \int_Z |H(t, z)|^p \nu(dz) dt \right] \right\}, \end{aligned} \quad (2.6)$$

where $H \in \mathcal{P}_2(t, E)$ and

$$I(t) = \int_0^t \int_Z H(s, z) \tilde{N}(dz, ds).$$

$\mathcal{P}_2(T, E)$ denotes the set of all equivalence classes of mappings $F : [0, T] \times E \times \Omega \rightarrow \mathbb{R}$ which coincide almost everywhere with respect to $\rho \times P$ and which satisfy the following conditions (see Page 225 of [1])

- (i) F is \mathbb{F} -predictable;
- (ii) $P \left(\int_0^T \int_E |F(t, x)|^2 \rho(dt, dx) < \infty \right) = 1$.

Now we are in the position to show our main result.

Theorem 2.3 *Let $g_1 : Z \times \Omega \rightarrow \mathbb{R}$ be measurable and fulfil the following*

$$\mathbb{E} \left[\left(\int_Z |g_1(z)|^2 \nu(dz) \right)^{p_0/2} + \int_Z |g_1(z)|^{p_0} \nu(dz) \right] < \infty$$

for some constant $p_0 > 2$. Suppose that the function g satisfies that

$$|g(t, x, z) - g(s, y, z)| \leq C_g \max \left\{ (t-s)^{\frac{1}{2}}, |x-y| \right\}^\beta g_1(z), \quad \text{for all } z \in Z, \text{ a.s.}, \quad (2.7)$$

and $g(0, 0, z) = 0$ uniformly for $z \in Z$ almost surely. Assume further that there exist positive constants γ_i ($i = 1, 2$) such that the non-random kernel function satisfies that for any $t \in (0, T]$,

$$\begin{aligned} \int_0^s \left(\int_{\mathbb{R}^d} |K(t-r, z) - K(s-r, z)| (1 + |z|^\beta) dz \right)^p dr &\leq N(T, \beta) (t-s)^{\frac{\gamma_1 p}{2}}, \\ \int_0^s \left(\int_{\mathbb{R}^d} |K(s-r, z)| dz \right)^p dr &\leq N_0, \\ \int_s^t \left(\int_{\mathbb{R}^d} |K(t-r, z)| (1 + |z|^\beta) dz \right)^p dr &\leq N(T, \beta) (t-s)^{\frac{\gamma_2 p}{2}}, \end{aligned}$$

where N_0 is a positive constant. Let D be an A -type bounded domain in \mathbb{R}^{d+1} such that $\bar{D} \subset \mathcal{O}_T$. Then we have, for $2 \leq p \leq p_0$ and $\beta < \alpha$,

$$[\mathcal{K}g(t, x)]_{\mathcal{L}^{p, \theta}(D; \delta)} \leq N(N_0, \beta, T, d, p),$$

where $\theta = 1 + \frac{\gamma p}{d+2}$ and $\gamma = \min\{\gamma_1, \gamma_2, \beta\}$.

Proof. Similar to the proof of Theorem 2.1 and using the inequality (2.6) we first have the following estimates.

$$\begin{aligned} & \mathbb{E} |\mathcal{G}g(t, x) - \mathcal{G}g(s, y)|^p \\ &= \mathbb{E} \left[\left| \int_0^t \int_Z \int_{\mathbb{R}^d} K(t-r, \xi) g(r, x-\xi, z) d\xi \tilde{N}(dz, dr) \right. \right. \\ & \quad \left. \left. - \int_0^s \int_Z \int_{\mathbb{R}^d} K(s-r, \xi) g(r, y-\xi, z) d\xi \tilde{N}(dz, dr) \right|^p \right] \\ &\leq N(p) \mathbb{E} \left[\left| \int_0^s \int_Z \int_{\mathbb{R}^d} [K(t-r, \xi) - K(s-r, \xi)] g(r, x-\xi, z) d\xi \tilde{N}(dz, dr) \right. \right. \\ & \quad \left. \left. + \int_0^s \int_Z \int_{\mathbb{R}^d} K(s-r, \xi) [g(r, x-\xi, z) - g(r, y-\xi, z)] d\xi \tilde{N}(dz, dr) \right. \right. \\ & \quad \left. \left. + \int_s^t \int_Z \int_{\mathbb{R}^d} K(t-r, \xi) g(r, x-\xi, z) d\xi \tilde{N}(dz, dr) \right|^p \right] \\ &\leq N(p) \mathbb{E} \left[\left(\int_s^t \int_Z \left| \int_{\mathbb{R}^d} K(t-r, \xi) g(r, x-\xi, z) d\xi \right|^2 \nu(dz) dr \right)^{p/2} \right] \\ & \quad + N(p) \mathbb{E} \left[\int_s^t \int_Z \left| \int_{\mathbb{R}^d} K(t-r, \xi) g(r, x-\xi, z) d\xi \right|^p \nu(dz) dr \right] \\ & \quad + N(p) \mathbb{E} \left[\left(\int_0^s \int_Z \left| \int_{\mathbb{R}^d} K(s-r, \xi) [g(r, x-\xi, z) - g(r, y-\xi, z)] d\xi \right|^2 \nu(dz) dr \right)^{p/2} \right] \\ & \quad + N(p) \mathbb{E} \left[\int_0^s \int_Z \left| \int_{\mathbb{R}^d} K(s-r, \xi) [g(r, x-\xi, z) - g(r, y-\xi, z)] d\xi \right|^p \nu(dz) dr \right] \\ & \quad + N(p) \mathbb{E} \left[\left(\int_0^s \int_Z \left| \int_{\mathbb{R}^d} [K(t-r, \xi) - K(s-r, \xi)] g(r, x-\xi, z) d\xi \right|^2 \nu(dz) dr \right)^{p/2} \right] \\ & \quad + N(p) \mathbb{E} \left[\int_0^s \int_Z \left| \int_{\mathbb{R}^d} [K(t-r, \xi) - K(s-r, \xi)] g(r, x-\xi, z) d\xi \right|^p \nu(dz) dr \right]. \end{aligned}$$

By using (2.7) and $g(0, 0, z) = 0$ uniformly for $z \in Z$ almost surely, we have that the above inequality is smaller than or equal to

$$\begin{aligned} & N \mathbb{E} \left[\left(\int_s^t \int_Z g_1(z)^2 \left| \int_{\mathbb{R}^d} |K(t-r, \xi)| (|x-x_0|^\beta + |x_0-\xi|^\beta) d\xi \right|^2 \nu(dz) dr \right)^{p/2} \right] \\ & + N \mathbb{E} \left[\int_s^t \int_Z |g_1(z)|^p \left| \int_{\mathbb{R}^d} |K(t-r, \xi)| (|x-x_0|^\beta + |x_0-\xi|^\beta) d\xi \right|^p \nu(dz) dr \right] \\ & + N \mathbb{E} \left[\left(\int_0^s \int_Z g_1(z)^2 \left| \int_{\mathbb{R}^d} |K(s-r, \xi)| |x-y|^\beta d\xi \right|^2 \nu(dz) dr \right)^{p/2} \right] \\ & + N \mathbb{E} \left[\int_0^s \int_Z |g_1(z)|^p \left| \int_{\mathbb{R}^d} |K(s-r, \xi)| |x-y|^\beta d\xi \right|^p \nu(dz) dr \right] \\ & + N \mathbb{E} \left[\left(\int_0^s \int_Z g_1(z)^2 \left| \int_{\mathbb{R}^d} |K(t-r, \xi) - K(s-r, \xi)| (|x-x_0|^\beta + |x_0-\xi|^\beta) d\xi \right|^2 \nu(dz) dr \right)^{p/2} \right] \\ & + N \mathbb{E} \left[\int_0^s \int_Z |g_1(z)|^p \left| \int_{\mathbb{R}^d} |K(t-r, \xi) - K(s-r, \xi)| (|x-x_0|^\beta + |x_0-\xi|^\beta) d\xi \right|^p \nu(dz) dr \right]. \quad (2.8) \end{aligned}$$

Following the proof of Theorem 2.1, we have

$$0 \leq t - s \leq 2c^2 \quad \text{and} \quad |x - y| \leq |x - x_0| + |y - x_0| \leq 2c.$$

Thus (2.8) yields that

$$\begin{aligned} & \mathbb{E}|\mathcal{G}g(t, x) - \mathcal{G}g(s, y)|^p \\ & \leq N(p, T, |x_0|)(1 + c^{\beta p})\mathbb{E} \left[\left(\int_s^t \int_Z g_1(z)^2 \left| \int_{\mathbb{R}^d} |K(t-r, \xi)|(1 + |\xi|^\beta) d\xi \right|^2 \nu(dz) dr \right)^{p/2} \right] \\ & \quad + N(p, T)(1 + c^{\beta p})\mathbb{E} \left[\int_s^t \int_Z |g_1(z)|^p \left| \int_{\mathbb{R}^d} |K(r, \xi)|(1 + |\xi|^\beta) d\xi \right|^p \nu(dz) dr \right] \\ & \quad + N(p, T)c^{\beta p}\mathbb{E} \left[\left(\int_0^s \int_Z g_1(z)^2 \left| \int_{\mathbb{R}^d} |K(r, \xi)| d\xi \right|^2 \nu(dz) dr \right)^{p/2} \right] \\ & \quad + N(p, T)c^{\beta p}\mathbb{E} \left[\int_0^s \int_Z |g_1(z)|^p \left| \int_{\mathbb{R}^d} |K(r, \xi)| d\xi \right|^p \nu(dz) dr \right] \\ & \quad + N(p, T)(1 + c^{\beta p})\mathbb{E} \left[\left(\int_0^s \int_Z g_1(z)^2 \left| \int_{\mathbb{R}^d} |K(t-r, \xi) - K(s-r, \xi)|(1 + |\xi|^\beta) d\xi \right|^2 \nu(dz) dr \right)^{p/2} \right] \\ & \quad + N(p, T)(1 + c^{\beta p})\mathbb{E} \left[\int_0^s \int_Z |g_1(z)|^p \left| \int_{\mathbb{R}^d} |K(t-r, \xi) - K(s-r, \xi)|(1 + |\xi|^\beta) d\xi \right|^p \nu(dz) dr \right] \\ & \leq N(\beta, p, T, N_0)[1 + c^{(1-\beta)p}](c^{\gamma_1 p} + c^{\gamma_2 p} + c^{\beta p}). \end{aligned}$$

Similar to the proof of Theorem 2.1, by using the properties of A -type domain, one can complete the proof of Theorem 2.3. \square

Corollary 2.2 *Assume all the assumptions in Theorem 2.3 hold, then*

$$\mathcal{G}g(t, x) \in C^\gamma((\bar{D}; \delta); L^p(\Omega)).$$

Remark 2.3 *In Theorem 2.3, both indices $\gamma_i, i = 1, 2$, depend on the parameter p . On the other hand, we notice that when $p = 2$, the two indices $\gamma_i, i = 1, 2$ will coincide with those in Theorem 2.1. It then follows from Proposition 2.1 that $p \geq 1$ is necessary and hence we can let $p = 2$. Moreover, γ will be largest in case $p = 2$.*

3 Applications

In this section, applying Theorems 2.1, 2.2 and 2.3, we give some examples.

3.1 Application to Parabolic Equations Driven by Brownian Motion

In this subsection, We first consider the following stochastic parabolic equations

$$\begin{cases} du(t, x) = (\Delta u + \operatorname{div} B(u) + c(t, x)u + f(t, x))dt + g(t, x)dW(t), & t > 0, x \in \mathbb{R}^d, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^d. \end{cases} \quad (3.1)$$

The existence and uniqueness of (3.1) has been obtained by many authors, see [6, 7]. Under the assumption the flux function B is continuous with linear growth. Debussche et al. [9] obtained the following results, see Theorem 2.5 in [8].

Proposition 3.1 *There exists $((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}), \tilde{W}, \tilde{u})$ which is a weak martingale solution to (3.1) and for all $p \in [2, \infty)$ and $u_0 \in L^p(\tilde{\Omega}; L^p)$,*

$$\tilde{u} \in L^p(\tilde{\Omega}; C([0, T]; L^2; L^2) \cap L^p(\tilde{\Omega}; L^\infty(0, T; L^p)) \cap L^p(\tilde{\Omega}; L^2(0, T; W^{1,2})).$$

Kim [20] obtained the Hölder estimate of (3.1), where they used Bessel space similar to those in [24, 19, 17]. Based on the theory of semigroup, Kuksin et al. [22] obtained the Hölder estimate of (3.1).

Let D be an A -type bounded domain in \mathbb{R}^{n+1} . Note that the Schauder estimate in this paper is interior estimate. It is well known that the solution of

$$u_t(t, x) = \Delta u + c(t, x)u + f(t, x)$$

has the interior Schauder estimate if c and f are Hölder continuous. Let v be the solution of the following stochastic heat equation

$$\begin{cases} du(t, x) = \Delta u dt + g(t, x) dW(t), & t > 0, x \in \mathbb{R}^d, \\ u(0, x) = 0, & x \in \mathbb{R}^d. \end{cases} \quad (3.2)$$

Set $w := u - v$, the w satisfies that

$$\begin{cases} w_t(t, x) = \Delta w + \operatorname{div} B(u) + c(t, x)u + f(t, x), & t > 0, x \in \mathbb{R}^d, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \quad (3.3)$$

Borrowing the idea from [8] and using the [8, Theorem 3.2], it is not hard to prove that the solution w of (3.3) is Hölder continuous. That is, there exists a positive constant γ such that

$$\mathbb{E} \|w\|_{C^\gamma(D_T)} = \mathbb{E} \sup_{t, x \in D_T} |u(t, x)| + \mathbb{E} \sup_{(t, x) \neq (s, y)} \frac{|u(t, x) - u(s, y)|}{\max\{|t - s|^{\frac{1}{2}}, |x - y|\}^\gamma} < \infty,$$

where $D_T = [0, T] \times G$ and G is a bounded domain in \mathbb{R}^d . Note that

$$\sup_{(t, x) \neq (s, y)} \frac{\mathbb{E} |u(t, x) - u(s, y)|}{\max\{|t - s|^{\frac{1}{2}}, |x - y|\}^\gamma} \leq \mathbb{E} \sup_{(t, x) \neq (s, y)} \frac{|u(t, x) - u(s, y)|}{\max\{|t - s|^{\frac{1}{2}}, |x - y|\}^\gamma},$$

we have the w of (3.3) belongs to $C^\gamma((\bar{D}_T; \delta); L^p(\Omega))$ for some $\gamma > 0$.

It is easy to see that the mild solution v of (3.2) takes the following form

$$v(t, x) = \mathcal{K}g(t, x) = \int_0^t \int_{\mathbb{R}^d} K(t, r, y) g(r, x - y) dy dW(r),$$

where $K(t, r; x, y) = (4\pi(t - r))^{-\frac{d}{2}} e^{-\frac{(x-y)^2}{4(t-r)}}$. It is easy to check that the kernel function K satisfies

$$\int_{\mathbb{R}^d} K(t, r; x) dx = 1, \quad \int_{\mathbb{R}^d} |x|^\beta K(t, r; x) dx \leq N(T) \quad \text{for } t \in [0, T],$$

which implies that (2.3) and (2.4) with $\gamma_2 = 1$ hold. Moreover, we have

$$\begin{aligned} & \int_0^s \left(\int_{\mathbb{R}^d} |K(t - r, z) - K(s - r, z)| (1 + |z|^\beta) dz \right)^2 dr \\ &= (t - s)^2 \int_0^s \left(\int_{\mathbb{R}^d} \left[\frac{d}{2(\xi - r)} - \frac{z^2}{4(\xi - r)^2} \right] (4\pi(\xi - r))^{-\frac{d}{2}} e^{-\frac{z^2}{4(\xi - r)}} (1 + |z|^\beta) dz \right)^2 dr \\ &\leq N(d, \beta) (t - s)^2 \int_0^s (\xi - r)^{-d-2} \left(\int_{\mathbb{R}^d} \left[1 + |z|^\beta + \frac{z^2}{4(\xi - r)} + \frac{|z|^{2+\beta}}{4(\xi - r)} \right] e^{-\frac{z^2}{4(\xi - r)}} dz \right)^2 dr \\ &\leq N(d, T, \beta) (t - s)^2 \int_0^s (\xi - r)^{-2} dr \\ &\leq N(d, T, \beta) (t - s)^2 [(\xi - s)^{-1} - \xi^{-1}] \\ &\leq N(d, T, \beta, \theta) (t - s), \end{aligned}$$

where $\xi = \theta t + (1-\theta)s$ and $\theta \in (0, 1)$. And thus (2.2) holds with $\gamma_1 = 1$. Therefore, the assumptions of Theorems 2.1 and 2.2 hold. It follows from Theorem 2.1 that

$$v(t, x) \in C^\beta((\bar{D}_T; \delta); L^p(\Omega)).$$

Combining the above results, we have the following

Theorem 3.1 *Let D_T be an A -type bounded domain in \mathbb{R}^{d+1} such that $D_T \subset \mathcal{O}_T$. Suppose the flux function B is continuous with linear growth, $u_0 \in C^\beta(\mathbb{R}^d)$ and $g \in C^\beta(\mathbb{R}_+ \times \mathbb{R}^d)$ with $g(0, 0) = 0$ almost surely, $0 < \beta < 1$, then the solution u of (3.1) is Hölder continuous in domain D_T .*

Similarly, we can use Theorem 2.2 to obtain the Schauder estimate of (3.1), where g does not depend on the time variable.

Next, we consider the following stochastic fractional heat equation

$$\begin{cases} du(t, x) = \Delta^{\frac{\alpha}{2}} u dt + g(t, x) dW(t), & t > 0, x \in \mathbb{R}^d, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \quad (3.4)$$

where $\Delta^{\frac{\alpha}{2}} := -(-\Delta)^{\frac{\alpha}{2}}$. Following the result of [30], the solution u of (3.4) can be written as

$$\begin{aligned} u(t, x) &= (\mathcal{G} * u_0)(t, x) + (\mathcal{G} * g)(t, x) \\ &= \int_{\mathbb{R}^d} p(t; x, y) u_0(y) dy + \int_0^t \int_{\mathbb{R}^d} p(t, r; x, y) g(r, y) dy dW(r), \end{aligned} \quad (3.5)$$

where the kernel function p has the following properties:

- for any $t > 0$,

$$\|p(t, \cdot)\|_{L^1(\mathbb{R}^d)} = 1 \text{ for all } t > 0.$$

- $p(t, x, y)$ is C^∞ on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ for each $t > 0$;
- for $t > 0$, $x, y \in \mathbb{R}^d$, $x \neq y$, the sharp estimate of $\widehat{p}(t, x)$ is

$$p(t, x, y) \approx \min \left(\frac{t}{|x - y|^{d+\alpha}}, t^{-d/\alpha} \right);$$

- for $t > 0$, $x, y \in \mathbb{R}^d$, $x \neq y$, the estimate of the first order derivative of $\widehat{p}(t, x)$ is

$$|\nabla_x p(t, x, y)| \approx |y - x| \min \left\{ \frac{t}{|y - x|^{d+2+\alpha}}, t^{-\frac{d+2}{\alpha}} \right\}.$$

The notation $f(x) \approx g(x)$ means that there is a number $0 < C < \infty$ independent of x , i.e. a constant, such that for every x we have $C^{-1}f(x) \leq g(x) \leq Cf(x)$.

Proposition 3.2 [30, Lemma 2.1] *For any $m \geq 0$, we have*

$$\partial_x^m p(t, x) = \sum_{n=0}^{n=\lfloor \frac{m}{2} \rfloor} C_n |x|^{m-2n} \min \left\{ \frac{t}{|x|^{d+\alpha+2(m-n)}}, t^{-\frac{d+2(m-n)}{\alpha}} \right\},$$

where $\lfloor \frac{m}{2} \rfloor$ means the largest integer that is less than $\frac{m}{2}$.

By using Proposition 3.2, we show the following

Lemma 3.1 *Let $0 \leq \epsilon < \frac{\alpha}{2}$. The following estimates hold.*

$$\begin{aligned} \int_0^s \left(\int_{\mathbb{R}^d} |\nabla^\epsilon p(t-r, z) - \nabla^\epsilon p(s-r, z)|(1+|z|^\beta) dz \right)^2 dr &\leq N(T, \beta)(t-s)^\gamma, \\ \int_0^s \left(\int_{\mathbb{R}^d} |\nabla^\epsilon p(s-r, z)| dz \right)^2 dr &\leq N_0, \\ \int_s^t \left(\int_{\mathbb{R}^d} |\nabla^\epsilon p(t-r, z)|(1+|z|^\beta) dz \right)^2 dr &\leq N(T, \beta)(t-s)^\gamma, \end{aligned}$$

where $\gamma = \frac{\alpha-2\epsilon}{\alpha}$.

Proof. For simplicity, we first prove the estimates with $\beta = 0$ hold. It is not hard to prove that when $\beta > 0$, the index will be improved and the proof is omitted here. Noting that $\partial_t p = -(-\Delta)^{\frac{\alpha}{2}} p := \nabla^\alpha p$, when $\lfloor \frac{\alpha+\epsilon}{2} \rfloor < 1$, we have

$$\begin{aligned} &\int_0^s \left(\int_{\mathbb{R}^d} |\nabla^\epsilon p(t-r, z) - \nabla^\epsilon p(s-r, z)| dz \right)^2 dr \\ &\leq (t-s)^2 \int_0^s \left(\int_{\mathbb{R}^d} |\nabla^{\alpha+\epsilon} p(\xi-r, z)| dz \right)^2 dr \\ &\leq (t-s)^2 \int_0^s \left(\int_{\mathbb{R}^d} |z|^{\alpha+\epsilon} \min \left\{ \frac{\xi-r}{|z|^{d+3\alpha+2\epsilon}}, (\xi-r)^{-\frac{d+2\alpha+2\epsilon}{\alpha}} \right\} dz \right)^2 dr \\ &\leq (t-s)^2 \int_0^s \left(\int_0^{(\xi-r)^{\frac{1}{\alpha}}} |z|^{\alpha+\epsilon} |z|^{d-1} (\xi-r)^{-\frac{d+2\alpha+2\epsilon}{\alpha}} d|z| \right. \\ &\quad \left. + \int_{(\xi-r)^{\frac{1}{\alpha}}}^\infty |z|^{\alpha+\epsilon} |z|^{d-1} \frac{\xi-r}{|z|^{d+3\alpha+2\epsilon}} d|z| \right)^2 dr \\ &\leq N(d, \alpha)(t-s)^2 \int_0^s (\xi-r)^{-2\frac{\alpha+\epsilon}{\alpha}} dr \\ &\leq N(d, \alpha, \theta)(t-s)^{\frac{\alpha-2\epsilon}{\alpha}}, \end{aligned}$$

where $\xi = \theta t + (1-\theta)s$ and $\theta \in (0, 1)$.

When $1 \leq \lfloor \frac{\alpha+\epsilon}{2} \rfloor < 2$, there is a little different from the above discussion. Similarly, we get

$$\begin{aligned} &\int_0^s \left(\int_{\mathbb{R}^d} |\nabla^\epsilon p(t-r, z) - \nabla^\epsilon p(s-r, z)| dz \right)^2 dr \\ &\leq (t-s)^2 \int_0^s \left(\int_{\mathbb{R}^d} |z|^{\alpha+\epsilon} \min \left\{ \frac{\xi-r}{|z|^{d+3\alpha+2\epsilon}}, (\xi-r)^{-\frac{d+2\alpha+2\epsilon}{\alpha}} \right\} dz \right)^2 dr \\ &\quad + (t-s)^2 \int_0^s \left(\int_{\mathbb{R}^d} |z|^{\alpha+\epsilon-2} \min \left\{ \frac{\xi-r}{|z|^{d+3\alpha+2\epsilon-2}}, (\xi-r)^{-\frac{d+2\alpha+2\epsilon-2}{\alpha}} \right\} dz \right)^2 dr \\ &\leq (t-s)^2 \int_0^s \left(\int_0^{(\xi-r)^{\frac{1}{\alpha}}} |z|^{d-1} |z|^{\alpha+\epsilon-2} \left[|z|^2 (\xi-r)^{-\frac{d+2\alpha+2\epsilon}{\alpha}} + (\xi-r)^{-\frac{d+2\alpha+2\epsilon-2}{\alpha}} \right] d|z| \right. \\ &\quad \left. + \int_{(\xi-r)^{\frac{1}{\alpha}}}^\infty |z|^{\alpha+\epsilon-2} |z|^{d-1} \left[|z|^2 \frac{\xi-r}{|z|^{d+3\alpha+2\epsilon}} + \frac{\xi-r}{|z|^{d+3\alpha+2\epsilon-2}} \right] d|z| \right)^2 dr \\ &\leq N(d, \alpha)(t-s)^2 \int_0^s (\xi-r)^{-2\frac{\alpha+\epsilon}{\alpha}} dr \\ &\leq N(d, \alpha, \theta)(t-s)^{\frac{\alpha-2\epsilon}{\alpha}}, \end{aligned}$$

where $\xi = \theta t + (1 - \theta)s$ and $\theta \in (0, 1)$.

Using Proposition 3.2 again, we have

$$\begin{aligned}
& \int_0^s \left(\int_{\mathbb{R}^d} |\nabla^\epsilon p(s-r, z)| dz \right)^2 dr \\
& \leq \int_0^s \left(\int_{\mathbb{R}^d} |z|^\epsilon \min \left\{ \frac{s-r}{|z|^{d+\alpha+2\epsilon}}, (s-r)^{-\frac{d+2\epsilon}{\alpha}} \right\} dz \right)^2 dr \\
& \leq \int_0^s \left(\int_0^{(s-r)^{\frac{1}{\alpha}}} |z|^\epsilon (s-r)^{-\frac{d+2\epsilon}{\alpha}} |z|^{d-1} d|z| \right. \\
& \quad \left. \int_{(s-r)^{\frac{1}{\alpha}}}^\infty |z|^\epsilon \frac{s-r}{|z|^{d+\alpha+2\epsilon}} |z|^{d-1} d|z| \right)^2 dr \\
& \leq N(d) \int_0^s (s-r)^{-\frac{2\epsilon}{\alpha}} dr \\
& \leq N(d, \alpha, \epsilon) s^{1-\frac{2\epsilon}{\alpha}} := N_0 < \infty.
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
& \int_s^t \left(\int_{\mathbb{R}^d} |\nabla^\epsilon p(t-r, z)|(1 + |z|^\beta) dz \right)^2 dr \\
& \leq N(d, \alpha, \epsilon) \int_s^t (t-r)^{-\frac{2\epsilon}{\alpha}} dr \\
& \leq N(d, \alpha, \epsilon) (t-s)^{1-\frac{2\epsilon}{\alpha}}.
\end{aligned}$$

The proof is complete. \square

Theorem 2.1 implies that the solution u of (3.5) satisfying $u \in C^{\epsilon+\beta_1, \beta_1/2}((\bar{D}; \delta); L^p(\Omega))$, where $\beta_1 = \min\{\beta, 2\gamma\}$.

Theorem 3.2 *Let D_T be a A -type bounded domain in \mathbb{R}^{d+1} such that $D_T \subset \mathcal{O}_T$. Suppose that $u_0 \in C^\beta(\mathbb{R}^d)$ and $g \in C^\beta(\mathbb{R}_+ \times \mathbb{R}^d)$ with $g(0, 0) = 0$ almost surely, $0 < \beta < 1$, then the solution u of (3.4) is Hölder continuous in domain D_T .*

Remark 3.1 *Comparing with Theorems 3.1 and 3.2, we find that if we take $\epsilon = 0$, then Theorem 3.2 with $\alpha = 2$ becomes Theorem 3.1. Let us compare the index of spatial variable. Theorem 3.1 shows that the index is β and Theorem 3.2 shows that the index is $\epsilon + \min\{\beta, 2\gamma\}$. When $\beta \leq 2\gamma$, the result of Theorem 3.2 is better than that of Theorem 3.2.*

3.2 Application to Fractional Heat Equations Driven by Lévy Noise

For simplicity, we consider the following SPDEs

$$\begin{cases} du(t, x) = \Delta^{\frac{\alpha}{2}} u(t, x) dt + \int_Z g(t, x, z) \tilde{N}(dt, dz), & t > 0, x \in \mathbb{R}^d, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \quad (3.6)$$

where $\Delta^{\frac{\alpha}{2}} = -(-\Delta)^{\frac{\alpha}{2}}$. The well-posedness of (3.6) has been proved by [17]. The solution of (3.6) can be written as

$$\begin{aligned}
u(t, x) &= (\mathcal{G} * u_0)(t, x) + (\mathcal{G} * g)(t, x) \\
&= \int_{\mathbb{R}^d} p(t; x, y) u_0(y) dy + \int_0^t \int_{\mathbb{R}^d} \int_Z p(t, r; x, y) g(r, y, z) dy \tilde{N}(dt, dz). \quad (3.7)
\end{aligned}$$

Using the properties of g and Lemma 3.1, it is easy to verify that all the assumptions in Theorem 2.3 hold for the kernel function.

Theorem 3.3 *Suppose that $u_0 \in C^\beta(\mathbb{R}^d)$ with $\beta < \alpha$ and the function g satisfies that*

$$|g(t, x, z) - g(s, y, z)| \leq C_g \max \left\{ (t - s)^{\frac{1}{2}}, |x - y| \right\}^\beta g_1(z), \quad \text{for all } z \in Z, \quad a.s.,$$

and $g(0, 0, z) = 0$ uniformly for $z \in Z$ almost surely, where there exists a constant $p_0 > 1$ such that $g_1(z)$ satisfies that

$$\mathbb{E} \left[\left(\int_Z |g_1(z)|^2 \nu(dz) \right)^{p_0/2} + \int_Z |g_1(z)|^{p_0} \nu(dz) \right] < \infty.$$

Let D be a A -type bounded domain in \mathbb{R}^{d+1} such that $\bar{D} \subset \mathcal{O}_T$. Then the solution u of (3.6) is Hölder continuous in domain D_T .

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A Appendix

Lemma A.1 *A Campanato space defined by Definition 2.1 is a Banach space. Moreover, if $1 \leq p \leq q < \infty$, $(\theta - p)/p \leq (\sigma - p)/q$, it holds that*

$$\mathcal{L}^{q,\sigma}(D; \delta) \subset \mathcal{L}^{p,\theta}(D; \delta).$$

Proof. Let $\{u^n\}_{n \geq 1}$ be a Cauchy sequence in $\mathcal{L}^{p,\theta}(D; \delta)$, then $\{u^n\}_{n \geq 1}$ is also a Cauchy in $L^p(D)$. Therefore, there is a measurable function $u \in L^p(D)$, such that

$$u^n \rightarrow u \text{ in } L^p(D).$$

By virtue of the Fato lemma, we have the following estimate:

$$\begin{aligned} [u - u^n]_{\mathcal{L}^{p,\theta}(D; \delta)}^p &= \sup_{X \in D, d \geq \rho > 0} \frac{1}{|D(X, \rho)|^\theta} \int_{D(X, \rho)} |(u - u^n)(Y) - (u - u^n)_{X, \rho}|^p dY \\ &= \sup_{X \in D, d \geq \rho > 0} \lim_m \frac{1}{|D(X, \rho)|^\theta} \int_{D(X, \rho)} |(u^m - u^n)(Y) - (u^m - u^n)_{X, \rho}|^p dY \\ &\leq \limsup_m \sup_{X \in D, d \geq \rho > 0} \frac{1}{|D(X, \rho)|^\theta} \int_{D(X, \rho)} |(u^m - u^n)(Y) - (u^m - u^n)_{X, \rho}|^p dY. \end{aligned}$$

To prove $u^n \rightarrow u$ in $\mathcal{L}^{p,\theta}(D; \delta)$, it suffices to show $[u]_{\mathcal{L}^{p,\theta}(D; \delta)} < \infty$. And this estimate holds clearly by using above calculation, so the first part of the lemma is finished.

To verify the second part, we use the Hölder inequality for $1 \leq p \leq q < \infty$, $0 < \theta, \sigma$ to gain that: for every $u \in \mathcal{L}^{q,\sigma}(D; \delta)$

$$\begin{aligned} [u]_{\mathcal{L}^{p,\theta}(D; \delta)} &= \left(\sup_{X \in D, d \geq \rho > 0} \frac{1}{|D(X, \rho)|^\theta} \int_{D(X, \rho)} |u(Y) - u_{X, \rho}|^p dY \right)^{1/p} \\ &\leq \left(\sup_{X \in D, d \geq \rho > 0} \frac{1}{|D(X, \rho)|^{\theta q/p + p - q}} \int_{D(X, \rho)} |u(Y) - u_{X, \rho}|^q dY \right)^{1/q} \\ &= \left(\sup_{X \in D, d \geq \rho > 0} \frac{1}{|D(X, \rho)|^\sigma} \frac{1}{|D(X, \rho)|^{\theta q/p + p - q - \sigma}} \int_{D(X, \rho)} |u(Y) - u_{X, \rho}|^q dY \right)^{1/q}. \end{aligned}$$

If $\theta q/p + p - q - \sigma \leq 0$, i.e. $(\theta - p)/p \leq (\sigma - p)/q$, then

$$[u]_{\mathcal{L}^{p,\theta}(D; \delta)} \leq C \left(\sup_{X \in D, d \geq \rho > 0} \frac{1}{|D(X, \rho)|^\sigma} \int_{D(X, \rho)} |u(Y) - u_{X, \rho}|^q dY \right)^{1/q},$$

for D is bounded.

On the other, $L^q(D) \subset L^p(D)$, we complete the proof. \square